AN INTRODUCTION TO REPRESENTATIONS OF HEREDITARY ALGEBRAS

HENNING KRAUSE

Overview

This is an extended abstract for a series of six lectures at the *12th Ring, Module and Algebra Workshop* (Afyon Kocatepe University, September 7-10, 2026). The topic of these lectures is the representation theory of hereditary algebras. More precisely, we consider finite dimensional associative algebras that are defined over a fixed field, and such an algebra is called *hereditary* if every submodule of a projective module is again projective. An equivalent condition says that $\operatorname{Ext}^n(X, Y) = 0$ for all pairs of modules X, Y and $n > 1$. Note that heredity is left-right symmetric. Thus an algebra is right hereditary if and only if it is left hereditary.

For an algebra A we are interested in its *representations*, so we look at the category of (right) A-modules. Our prime focus is on the full subcategory $\operatorname{mod} A$ of finitely presented A-modules. If A is finite dimensional over a field k, then the following are equivalent for an A-module X.

(1) X is finitely presented.
(2) X is finitely generated.
(3) X is noetherian.
(4) X is artinian.
(5) X has finite length.
(6) X is finite dimensional over k.

Thus it seems natural to focus on the category of modules satisfying these finiteness conditions.

In the following we give for each lecture a brief description of its content and include some references. The choice of topics depends on personal taste, of course, but we try to cover a variety of aspects, including also historical perspectives. The overall goal is to provide an introduction to the representation theory of hereditary algebras. The first two lectures offer some motivation and explain the basic structure of the category of finite dimensional representations. The subsequent four lectures are fairly independent from each other, and each of them is devoted to a particular topic or result.

1. From quivers to representations of hereditary algebras

A *quiver* Q is nothing but an oriented graph, and we may study its k-linear representations by assigning to each vertex a vector space and to each arrow a linear map. The k-linear representations of Q identify with modules over the *path algebra* kQ. A path algebra is hereditary, and it is finite dimensional if the quiver is finite without oriented cycles. A celebrated result of Gabriel [5, 6] describes the quivers that admit only finitely many isomorphism classes of indecomposable representations; they are precisely the quivers such that the underlying diagram is a disjoint union of Dynkin diagrams of type A_n, D_n, or E_6, E_7, E_8. A finite quiver

Q gives rise to the quadratic form

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha\colon i \to j} x_i x_j,$$

and taking a representation to its dimension vector yields a bijective correspondence between the isomorphism classes of indecomposable representations and the positive roots of the form, provided the quiver is of finite representation type. This will be illustrated by some examples. Further examples show that there are hereditary algebras which are not isomorphic or Morita equivalent to any path algebra, provided the base field is not algebraically closed.

2. The representation type via Auslander-Reiten theory

For any finite dimensional algebra A over a field k, Auslander and Reiten introduced what is now called the *Auslander-Reiten translate*. When A is hereditary it can be described by the functor

$$\tau = D\operatorname{Ext}^1_A(-,A)\colon \operatorname{mod} A \longrightarrow \operatorname{mod} A$$

where $D = \operatorname{Hom}_k(-,k)$ denotes the usual k-duality. The functor τ admits a left adjoint

$$\tau^- = \operatorname{Ext}^1_A(DA,-)\colon \operatorname{mod} A \longrightarrow \operatorname{mod} A.$$

By iterating these functors one defines the full subcategories

$$\mathcal{P} := \bigcup_{n \geq 0} \operatorname{Ker} \tau^n \qquad\qquad \mathcal{I} := \bigcup_{n \leq 0} \operatorname{Ker} \tau^n$$

of *preprojective* and *preinjective* modules. This yields a decomposition

$$\operatorname{mod} A = \mathcal{P} \vee \mathcal{R} \vee \mathcal{I}$$

where $\mathcal{R}$ denotes the subcategory of *regular* modules; it is a fundamental tool for the understanding of the representations of A. In particular, the representation type of A, which is either *finite*, *tame*, or *wild*, can be described in terms of this decomposition.

Useful references are the text books of Auslander, Reiten, and Smalø[3] and of Benson [4].

3. Thick subcategories and non-crossing partitions

Let A be a hereditary finite dimensional algebra. When A is a path algebra and of finite representation type, we have already seen a bijective correspondence between the isomorphism classes of indecomposable A-modules and the positive roots of a positive definite quadratic form q_A that is attached to A. There is actually a vast generalisation of this result which does not require any assumptions on A. The root-theoretic description of indecomposable modules in finite type has an analogue in which individual indecomposables are replaced by thick subcategories and positive roots are replaced by non-crossing partitions. Then one obtains a bijective correspondence between thick subcategories of $\operatorname{mod} A$ that are generated by a rigid module and non-crossing partitions in the Weyl group W_A. Here, a full additive subcategory $\mathcal{C} \subseteq \operatorname{mod} A$ is *thick* if it is closed under direct summands and for any short exact sequence $0 \to X \to Y \to Z \to 0$ the *two-out-of-three property* holds, so if two of X, Y, Z are in $\mathcal{C}$ then all three objects are in $\mathcal{C}$. An A-module X is *rigid* if $\operatorname{Ext}^1_A(X,X) = 0$. And the *non-crossing partitions* are the elements lying between the identity and a Coxeter element in the absolute order on W_A, which is actually a crystallographic Coxeter group. The correspondence is based on a careful analysis of the Grothendieck group $K_0(A)$, which comes equipped with a non-degenerate bilinear form, also known as Euler form.

The correspondence is due to Ingalls and Thomas for path algebras of quivers [9], and for the general case see [7].

4. Preprojective algebras and a theorem of Serre

For a hereditary finite dimensional algebra A the left adjoint τ^- of the Auslander-Reiten translate yields the *preprojective algebra*; it is by definition the $\mathbb{Z}$-graded orbit algebra

$$\Pi := \bigoplus_{n\geq 0} \operatorname{Hom}_A(A, \tau^{-n}A).$$

There is a hereditary abelian category $\mathcal{H}$ which one may view as the category of sheaves on a noncommutative curve that is attached to A. More precisely, we view $\operatorname{mod} A$ as a subcategory of its bounded derived category $\mathbf{D}^b(\operatorname{mod} A)$, and there is a distinguished t-structure which yields

$$\mathcal{H} := \mathcal{I}[-1] \vee \mathcal{P} \vee \mathcal{R}$$

as its heart. For example, when A equals the Kronecker algebra $\left[\begin{smallmatrix} k & 0 \\ k^2 & k \end{smallmatrix}\right]$, then $\mathcal{H}$ identifies with the category of coherent sheaves on the projective line over k.

Now consider for $r > 0$ the projective space $\mathbb{P}^r(k)$ of dimension r over an algebraically closed field k. A classical theorem of Serre [14] describes the category of coherent sheaves $\operatorname{coh} \mathbb{P}^r(k)$ in terms of $\mathbb{Z}$-graded modules over the orbit algebra

$$\Lambda := \bigoplus_{n\geq 0} \operatorname{Hom}_A(\mathscr{O}, \mathscr{O}(n)) \cong k[t_0, \dots, t_r].$$

More precisely, there is an equivalence

$$\operatorname{coh} \mathbb{P}^r(k) \simeq (\operatorname{grmod} \Lambda)/(\operatorname{grmod}_0 \Lambda),$$

where $\operatorname{grmod}_0 \Lambda$ denotes the subcategory of torsion modules. The analogue of Serre's theorem for A describes the abelian category $\mathcal{H}$ in terms of $\mathbb{Z}$-graded modules over the preprojective algebra Π, so

$$\mathcal{H} \simeq (\operatorname{grmod} \Pi)/(\operatorname{grmod}_0 \Pi).$$

The reference for this analogue of Serre's theorem is [10], but it goes back to work of Lenzing [12]. The abstract version of Serre's theorem uses categories of coherent functors in the sense of Auslander [2].

5. Counting subcategories of quiver representations

For any natural number n we consider representations of a linear oriented quiver of Dynkin type A_n, and its k-linear representations identify with modules over the algebra $T_n(k)$ of upper triangular $n \times n$ matrices over k. There are several types of additive subcategories of $\operatorname{mod} T_n(k)$, and counting them yields more ore less well known integer sequences. For instance, thick subcategories or torsion classes are counted by the Catalan number C_{n+1}, while quotient closed subcategories are counted by $(n+1)!$. In each case the inclusion relation between subcategories yields the structure of a complete lattice. It is interesting to note that the number of subcategories closed under extensions and direct summands produced a new entry in the On-Line Encyclopedia of Integer Sequences [1]; its identifier is A393920. Only recent progress involving AI provided further insight into the nature of this sequence.

The reference for this discussion of subcategories of $\operatorname{mod} T_n(k)$ and its combinatorics is [11].

6. Infinite dimensional representations of tame hereditary algebras

For a finite dimensional algebra A the usual focus is on its category of finite dimensional representations. When A is of finite representation type, this restriction is justified by the fact that an arbitrary A-module decomposes essentially uniquely into a direct sum of indecomposable modules, and all these indecomposables are actually finite dimensional. In fact, this property characterises the algebras of finite representation type. For that reason one may wonder about possible classifications of modules when A is of infinite representation type. There is a satisfactory answer for the class of pure-injective A-modules when A is of tame representation type. Note that the pure-injective modules are precisely the direct summands of a possibly infinite product of finite dimensional modules.

It was Ringel who initiated in [13] a systematic study of infinite dimensional modules, and we refer to [8] for a recent account which includes a complete description of all pure-injective A-modules when A is of tame type. In fact, it suffices to list all indecomposable pure-injective modules; their isomorphism classes provide the points of what is known as *Ziegler spectrum* of A. Of particular interest is the *generic module*. Assuming that A is connected, it is the unique endofinite module of finite length, and it is also the unique inedcomposable torsion-free and divisible A-module.

References

[1] The on-line encyclopedia of integer sequences. Published electronically at `https://oeis.org`, 2026.

[2] Maurice Auslander. Coherent functors. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 189–231. Springer-Verlag New York, Inc., New York, 1966.

[3] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[4] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991. Basic representation theory of finite groups and associative algebras.

[5] Peter Gabriel. Unzerlegbare Darstellungen. I. *Manuscripta Math.*, 6:71–103; correction, ibid. 6 (1972), 309, 1972. `doi:10.1007/BF01298413`.

[6] Peter Gabriel. Indecomposable representations. II. In *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971 & Convegno di Geometria, INDAM, Rome, 1972)*, pages 81–104. Academic Press, London-New York, 1973.

[7] Andrew Hubery and Henning Krause. A categorification of non-crossing partitions. *J. Eur. Math. Soc. (JEMS)*, 18(10):2273–2313, 2016. `doi:10.4171/JEMS/641`.

[8] Lidia Angeleri Hügel, Andrew Hubery, and Henning Krause. From finite to infinite length modules over tame hereditary algebras, 2026. `arXiv:2604.27528`.

[9] Colin Ingalls and Hugh Thomas. Noncrossing partitions and representations of quivers. *Compos. Math.*, 145(6):1533–1562, 2009. `doi:10.1112/S0010437X09004023`.

[10] Henning Krause. Serre's theorem for coherent sheaves via Auslander's techniques. *J. Noncommut. Geom.*, 20(1):209–220, 2026. `doi:10.4171/jncg/588`.

[11] Henning Krause and Balduin Stoye. Multisets of finite intervals and a universal category of poset representations, 2026. `arXiv:2601.22649`.

[12] Helmut Lenzing. Curve singularities arising from the representation theory of tame hereditary algebras. In *Representation theory, I (Ottawa, Ont., 1984)*, volume 1177 of *Lecture Notes in Math.*, pages 199–231. Springer, Berlin, 1986. `doi:10.1007/BFb0075266`.

[13] Claus Michael Ringel. Infinite-dimensional representations of finite-dimensional hereditary algebras. In *Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977)*, pages 321–412. Academic Press, London-New York, 1979.

[14] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Ann. of Math. (2)*, 61:197–278, 1955. `doi:10.2307/1969915`.

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany