QUANTUM GROUPS, LECTURE II: FROM REPRESENTATIONS TO QUANTUM FUNCTIONS: TANNAKA–KREIN AND $O_q(SL_2)$

Theme and objective

Lecture II takes the dual, representation-theoretic route. For a Hopf algebra H, finite-dimensional representations have matrix coefficients, and the tensor product of representations becomes multiplication of those coefficients. For $H = U_q(\mathfrak{sl}_2)$, the coefficients of the fundamental two-dimensional module generate $O_q(SL_2)$. The central calculation is that the invariant quantum exterior line in $V \otimes V$ yields the quantum determinant $ad - qbc = 1$.

The guiding passage is therefore $U_q(\mathfrak{sl}_2) \to \mathrm{Rep}_{\mathrm{fd}}(U_q(\mathfrak{sl}_2)) \to O_q(SL_2)$. For the explicit SL_2 theory see Kassel [Kassel 1995, Ch. VII, §§4–5, pp. 150–155] and Klimyk–Schmüdgen [Klimyk–Schmüdgen 1997, §§4.1–4.4, pp. 97–123]; for the general coordinate-algebra viewpoint see Lusztig [Lusztig 2010, §29.5].

Classical warm-up: the coordinate Hopf algebra of SL_2

Before quantizing anything, consider

$$SL_2 = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} : x_{11}x_{22} - x_{12}x_{21} = 1 \right\}.$$

Write the coordinate functions as

$$a(g) = g_{11}, \qquad b(g) = g_{12}, \qquad c(g) = g_{21}, \qquad d(g) = g_{22}.$$

Then $O(SL_2) = \Bbbk[a, b, c, d]/(ad - bc - 1)$. The group multiplication $m : SL_2 \times SL_2 \longrightarrow SL_2$ dualizes to a coproduct $\Delta : O(SL_2) \longrightarrow O(SL_2) \otimes O(SL_2)$. If

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the complete formula is simply $\Delta(T) = T \dot{\otimes} T$, where matrix multiplication is performed using tensor products of entries.

Equivalently, $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$. The counit is evaluation at the identity:

$$\varepsilon(a) = \varepsilon(d) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0.$$

The antipode is induced by matrix inversion, $S(T) = T^{-1}$.

The important point is not the formulas themselves.

It is that group multiplication $\leftrightarrow$ coproduct of coordinate functions. This is the literal affine-algebraic meaning of the Hopf algebra axioms. More generally, this is the commutative half of the classical Hopf dictionary from Lecture I: over $\Bbbk$, commutative Hopf algebras are contravariantly equivalent to affine group schemes. See Waterhouse [Waterhouse 1979, §§1.1–1.5]. The neutral Tannakian theorem below will recover precisely such an affine group scheme from a symmetric tensor category and its fiber functor.

Matrix coefficients

Let H be an algebra and let $\rho : H \longrightarrow \mathrm{End}_{\Bbbk}(V)$ be a finite-dimensional representation.

For $\varphi \in V^*, \qquad v \in V$, define the corresponding *matrix coefficient* $c_{\varphi,v} : H \longrightarrow \Bbbk$ by $c_{\varphi,v}(h) = \varphi(hv)$. If $v_1, \ldots, v_n$ is a basis of V, with dual basis $v_1^*, \ldots, v_n^*$, write $c_{ij} = c_{v_i^*, v_j}$. Then $hv_j = \sum_i c_{ij}(h) v_i$. Thus the c_{ij} are literally the entries of the matrix $\rho(h)$.

The finite dual. For an infinite-dimensional algebra H, its full linear dual H^* generally does not inherit a useful coalgebra structure from multiplication on H.

One therefore uses the *finite dual* $H^\circ = \{f \in H^* : f \text{ vanishes on some ideal of finite codimension}\}$. Every matrix coefficient of a finite-dimensional representation belongs to H°, because it factors through the finite-dimensional quotient $H/\ker\rho$. If H is a bialgebra, then H° is again a bialgebra; if H is a Hopf algebra, then H° is a Hopf algebra.

See Sweedler [Sweedler 1969, Ch. VI] and Kassel [Kassel 1995, Ch. III].

Coalgebra structure of matrix coefficients. The multiplication of H determines the coproduct on H°.

For the matrix coefficients c_{ij}, $c_{ij}(xy) = \sum_k c_{ik}(x) c_{kj}(y)$. Indeed, $yv_j = \sum_k c_{kj}(y) v_k$, so

$$xyv_j = \sum_{i,k} c_{ik}(x) c_{kj}(y) v_i.$$

Consequently, $\Delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$. Also, $\varepsilon(c_{ij}) = c_{ij}(1) = \delta_{ij}$. Thus every finite-dimensional representation naturally produces a matrix coalgebra.

Compare this with the coordinate functions on SL_2: it is exactly the same formula.

The central calculation: tensor products become products

Now suppose H is a bialgebra.

The multiplication on the dual is convolution: $(fg)(h) = f(h_{(1)}) g(h_{(2)})$. Let V, W be finite-dimensional H-modules. Choose $\varphi \in V^*, \quad v \in V, \quad \psi \in W^*, \quad w \in W$. Then

$$\begin{aligned} c_{\varphi,v}(h_{(1)}) c_{\psi,w}(h_{(2)}) &= \varphi(h_{(1)} v)\, \psi(h_{(2)} w) \\ &= (\varphi \otimes \psi)\left(h_{(1)} v \otimes h_{(2)} w\right) \\ &= (\varphi \otimes \psi)\left(h(v \otimes w)\right). \end{aligned}$$

Therefore $c_{\varphi,v}\, c_{\psi,w} = c_{\varphi\otimes\psi,\, v\otimes w}$. This deserves to remain on the board for the rest of the lecture: multiplication of functions $=$ tensor product of representations.

Proposition 0.1 (Matrix-coefficient principle)**.** *Let H be a bialgebra. The linear span of the matrix coefficients of all finite-dimensional H-modules is a sub-bialgebra $\mathcal{R}(H) \subseteq H^\circ$. If H is a Hopf algebra and the class of representations is closed under duals, then $\mathcal{R}(H)$ is a Hopf subalgebra.*

Moreover, $c^V_{\varphi,v}\, c^W_{\psi,w} = c^{V\otimes W}_{\varphi\otimes\psi,\, v\otimes w}$.

Proof. Closure under the coproduct follows from $\Delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$. Closure under multiplication is the tensor-product calculation just given.

The unit is the matrix coefficient of the trivial representation.

For a Hopf algebra, dual representations are defined using the antipode. This implies that $S(c)(h) = c(S(h))$ is again a matrix coefficient. □

This proposition is the elementary algebraic heart of Tannaka–Krein duality.

Tannaka–Krein reconstruction

We now formulate the categorical idea.

Suppose that C is a suitable rigid $\Bbbk$-linear monoidal category and $\omega : C \longrightarrow \mathrm{Vect}_{\Bbbk}^{\mathrm{fd}}$ is a faithful monoidal functor.

The functor ω forgets the representation structure and remembers only the underlying vector spaces.

The reconstruction problem asks:

> Can one recover the algebraic symmetry object from (C, ω)?

Under appropriate hypotheses, the answer is yes.

The coend. The coordinate-side reconstruction is naturally expressed as the coend

$$H_C = \int^{X \in C} \omega(X)^* \otimes \omega(X).$$

One may think of this as taking

$$\bigoplus_{X \in C} \omega(X)^* \otimes \omega(X)$$

and imposing the relations expressing naturality with respect to morphisms in C.

An elementary tensor $\varphi \otimes v$ is interpreted as an abstract matrix coefficient.

The tensor product of objects produces multiplication: $[\varphi \otimes v]_X \, [\psi \otimes w]_Y = [(\varphi \otimes \psi) \otimes (v \otimes w)]_{X \otimes Y}$. The matrix-coalgebra formula gives the coproduct.

If $e_1, \dots, e_n$ is a basis of $\omega(X)$, then $\Delta([\varphi \otimes v]) = \sum_i [\varphi \otimes e_i] \otimes [e_i^* \otimes v]$. Rigidity, i.e. the existence of dual objects, produces the antipode.

Thus:

categorical structure	reconstructed structure
objects and morphisms	coalgebra
tensor product	multiplication
tensor unit	unit
dual objects	antipode

This is perhaps the most economical conceptual explanation of why Hopf algebras occur whenever one studies tensor categories of representations.

For reconstruction in this form see Majid [Majid 1995, Ch. 9, especially §9.4] and Etingof–Gelaki–Nikshych–Ostrik [Etingof–Gelaki–Nikshych–Ostrik 2015, Ch. 5].

The classical Tannakian theorem. In the symmetric situation one recovers an ordinary affine group scheme.

Theorem 0.2 (Neutral Tannaka duality, schematic form)**.** *Let C be a rigid abelian symmetric tensor category over $\Bbbk$, assume $\mathrm{End}_C(\mathbf{1}) = \Bbbk$, and let $\omega : C \longrightarrow \mathrm{Vect}_\Bbbk^{\mathrm{fd}}$ be an exact faithful $\Bbbk$-linear tensor functor.*

Then the tensor automorphism functor $R \longmapsto \mathrm{Aut}^{\otimes}(\omega \otimes_\Bbbk R)$ is represented by an affine group scheme G, and $C \simeq \mathrm{Rep}_\Bbbk(G)$ as symmetric tensor categories.

See Deligne–Milne [Deligne–Milne 1982, Thm. 2.11].

The symmetry of the category is important.

For quantum groups the relevant representation category is generally not symmetric. One reconstructs a possibly noncommutative Hopf algebra instead.

This is the point at which the word "quantum" becomes categorical: one retains tensor products but gives up ordinary symmetry.

Cartan datum and the representation-theoretic input

Fix the finite-type Cartan/root datum from Lecture I. Its weight lattice X has dominant cone X^+, and for generic q the dominant weights $\lambda \in X^+$ parametrize the irreducible finite-dimensional type–1 modules $V(\lambda)$. The reason the algebraic q-form is particularly natural here is that the toral elements K_μ, $\mu \in Y$, are already part of the Hopf algebra and act on a weight vector v_λ by $K_\mu v_\lambda = q^{\langle \mu, \lambda \rangle} v_\lambda$. Thus the weight lattice is encoded multiplicatively inside the algebra, rather than through formal power-series exponentials. This is the representation-theoretic advantage of the q-form emphasized in Lecture I.

Definition 0.3. The quantized coordinate algebra $\mathcal{O}_q(G)$ is the Hopf subalgebra of the finite dual $U_q(\mathfrak{g})^\circ$ spanned by matrix coefficients of finite-dimensional type–1 $U_q(\mathfrak{g})$-modules.

Thus the enveloping and coordinate versions are linked by representation theory rather than by naive vector-space duality. See Lusztig [Lusztig 2010, §29.5].

Type A_1: the fundamental representation

For SL_2, let $X = \mathbb{Z}\omega$, $\alpha = 2\omega$. The fundamental dominant weight is ω. Let $V = V(\omega)$ have basis v_+, v_- with $Kv_+ = qv_+$, $Kv_- = q^{-1}v_-$, $Ev_+ = 0$, $Ev_- = v_+$, $Fv_+ = v_-$, and $Fv_- = 0$.

Define four matrix coefficients $a, b, c, d \in U_q(\mathfrak{sl}_2)^\circ$ by

$$\rho(h) = \begin{pmatrix} a(h) & b(h) \\ c(h) & d(h) \end{pmatrix}$$

with respect to the ordered basis v_+, v_-. Hence $a(h) = v_+^*(hv_+)$, $b(h) = v_+^*(hv_-)$, $c(h) = v_-^*(hv_+)$, and $d(h) = v_-^*(hv_-)$.

The coalgebra formulas are forced by matrix multiplication:

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Explicitly, $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = c \otimes a + d \otimes c$, and $\Delta(d) = c \otimes b + d \otimes d$.

Compare these formulas with the classical coordinate algebra of SL_2: the coalgebra formulas look identical.

The deformation appears in the multiplication relations among a, b, c, d. This is the dual face of the formal-moduli picture from Lecture I. On the enveloping side one naturally notices the deformation of comultiplication, because $U(\mathfrak{g})$ loses cocommutativity. On the coordinate side one naturally notices the deformation of multiplication, because $\mathcal{O}(G)$ loses commutativity while the matrix-coalgebra formulas may remain unchanged. Schematically,

$U(\mathfrak{g}) \rightsquigarrow U_q(\mathfrak{g})$	$\mathcal{O}(G) \rightsquigarrow \mathcal{O}_q(G)$
visible change in Δ	visible change in m
loss of cocommutativity	loss of commutativity.

This is a statement about convenient presentations, not a claim that only one structure map is intrinsically deforming. Hopf duality exchanges multiplication and comultiplication, so the two columns are dual descriptions of the same quantum deformation.

Where the quantum determinant comes from

This is the calculation I would emphasize most strongly at the board.

Consider $V \otimes V$. With our coproduct convention, $E(v_+ \otimes v_-) = q\, v_+ \otimes v_+$, whereas $E(v_- \otimes v_+) = v_+ \otimes v_+$. Therefore $w_q = v_+ \otimes v_- - q\, v_- \otimes v_+$ satisfies $Ew_q = 0$. Similarly, $Fw_q = 0$. Since $Kw_q = w_q$, the line $\Bbbk w_q \subseteq V \otimes V$ is the trivial $U_q(\mathfrak{sl}_2)$-module.

Thus $hw_q = \varepsilon(h)w_q \qquad (h \in U_q(\mathfrak{sl}_2))$. This one-dimensional subrepresentation is the quantum analogue of $\Lambda^2(\Bbbk^2)$. Classically, the fact that $\Lambda^2(\Bbbk^2)$ is the trivial representation of SL_2 produces the determinant relation.

Exactly the same phenomenon occurs quantum mechanically.

Extracting a matrix coefficient. Take the coefficient of $v_+ \otimes v_-$ in hw_q. For the first term, $h(v_+ \otimes v_-) = h_{(1)}v_+ \otimes h_{(2)}v_-$, and the coefficient of $v_+ \otimes v_-$ is $a(h_{(1)})d(h_{(2)}) = (ad)(h)$. For the second term, the coefficient of $v_+ \otimes v_-$ in $h(v_- \otimes v_+)$ is $b(h_{(1)})c(h_{(2)}) = (bc)(h)$. Hence $(ad - qbc)(h) = \varepsilon(h)$. Since the unit of the finite dual is the counit of U_q, we obtain $ad - qbc = 1$. This is the quantum determinant relation.

It has not been guessed.

It is the matrix coefficient of the statement $\Lambda^2_q V \cong \mathbf{1}$. That is the Tannakian meaning of the quantum determinant.

A second relation from the same vector. Now take the coefficient of $v_+ \otimes v_+$ in hw_q. Because the right-hand side $\varepsilon(h)w_q$ has no $v_+ \otimes v_+$ component, we obtain $(ab - qba)(h) = 0$ for every h.

Therefore $ab = qba$. Likewise the $v_- \otimes v_-$ coefficient gives $cd = qdc$. The remaining quadratic relations can be obtained by applying the same argument together with the dual module, or verified directly from the matrix coefficients.

The coordinate Hopf algebra $\mathcal{O}_q(SL_2)$

We now give the standard presentation compatible with our convention.

Definition 0.4. Let $q \in \Bbbk^\times$. The algebra $\mathcal{O}_q(SL_2)$ is generated by a, b, c, d, subject to $ab = qba$, $ac = qca$, $bd = qdb$, $cd = qdc$, $bc = cb$, $ad - da = (q - q^{-1})bc$, and the quantum determinant relation $ad - qbc = 1$. Equivalently, $da - q^{-1}bc = 1$.

For this presentation see Klimyk–Schmüdgen [Klimyk–Schmüdgen 1997, §4.1] and Kassel [Kassel 1995, Ch. VII, §4].

As always, some references use q^{-1} where we use q, or transpose the fundamental matrix. The present convention is chosen to match the coproduct convention of Lecture I and the invariant vector $v_+ \otimes v_- - q\, v_- \otimes v_+$.

Hopf structure. Put

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The coproduct is $\Delta(T) = T \dot{\otimes} T$.

The counit is

$$\varepsilon(a) = \varepsilon(d) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0.$$

The antipode is quantum matrix inversion:

$$S(T) = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}.$$

Indeed, $TS(T) = S(T)T = I$. For example, $a(-q^{-1}b) + ba = -q^{-1}ab + ba = 0$, and $ad - qbc = 1$. Thus $\mathcal{O}_q(SL_2)$ is a Hopf algebra.

Take-home theorem

Theorem 0.5 (Matrix coefficients reconstruct $\mathcal{O}_q(SL_2)$)**.** *Assume q is generic. Let $V = V(\omega)$ be the fundamental two-dimensional type–1 representation of $U_q(\mathfrak{sl}_2)$, with matrix coefficients a, b, c, d. The Hopf subalgebra of $U_q(\mathfrak{sl}_2)^\circ$ generated by these four coefficients is $\mathcal{O}_q(SL_2)$. Multiplication is induced by tensor product, the coproduct is $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$, and the invariant line $\Bbbk(v_+ \otimes v_- - q v_- \otimes v_+)$ yields $ad - qbc = 1$.*

Skeletal proof. Products of the four fundamental coefficients are precisely matrix coefficients of tensor powers of V. For generic q, every finite-dimensional irreducible type–1 $U_q(\mathfrak{sl}_2)$-module occurs in a suitable tensor power, so these coefficients generate the representative-function algebra. The matrix-coalgebra formula gives the coproduct, and the calculation with the invariant quantum exterior vector gives the determinant relation. See Kassel [Kassel 1995, Ch. VII, §§4–5] and Klimyk–Schmüdgen [Klimyk–Schmüdgen 1997, §§4.1 and 4.4]. □

Hopf duality and comodules

The inclusion $\mathcal{O}_q(SL_2) \subseteq U_q(\mathfrak{sl}_2)^\circ$ gives a bilinear pairing $\langle\, , \,\rangle : U_q(\mathfrak{sl}_2) \times \mathcal{O}_q(SL_2) \longrightarrow \Bbbk$. It satisfies $\langle xy, f\rangle = \langle x \otimes y, \Delta(f)\rangle$ and $\langle x, fg\rangle = \langle \Delta(x), f \otimes g\rangle$. Thus multiplication on one side is dual to comultiplication on the other.

For the fundamental matrix coefficients, $\langle K, a\rangle = q$, $\langle K, d\rangle = q^{-1}$, $\langle E, b\rangle = 1$, and $\langle F, c\rangle = 1$, with the other obvious generator pairings equal to zero.

For example, $\langle E, fg\rangle = \langle E \otimes 1 + K \otimes E, f \otimes g\rangle$. Hence $\langle E, fg\rangle = \langle E, f\rangle\varepsilon(g) + \langle K, f\rangle\langle E, g\rangle$. The twisted Leibniz rule from Lecture I has therefore reappeared, now on the dual side as the multiplication rule for coordinate functions.

Thus multiplication and coproduct are exchanged by the pairing, while modules on the enveloping side correspond to corepresentations on the coordinate side.

One should not write $\mathcal{O}_q(SL_2) = U_q(\mathfrak{sl}_2)^*$ literally: both Hopf algebras are infinite-dimensional.

The correct statement uses the finite dual, representative functions, or an appropriate Hopf pairing.

See Kassel [Kassel 1995, Ch. VII, §§4–5] and Klimyk–Schmüdgen [Klimyk–Schmüdgen 1997, §4.4].

Coordinate Hopf algebras are naturally expressed using corepresentations.

Let A be a coalgebra. A right A-comodule is a vector space V equipped with $\delta : V \longrightarrow V \otimes A$ satisfying $(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \Delta)\delta$ and $(\mathrm{id} \otimes \varepsilon)\delta = \mathrm{id}$. If $v_1, \dots, v_n$ is a basis and $\delta(v_j) = \sum_i v_i \otimes t_{ij}$, then coassociativity says precisely $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$. Thus a matrix $T = (t_{ij})$ whose entries satisfy $\Delta(T) = T \dot{\otimes} T$ is a corepresentation matrix.

This is the coordinate-side version of a representation.

Under a suitable Hopf pairing $H \times A \longrightarrow \Bbbk$, a right A-comodule produces a left H-module by $h \cdot v = v_{(0)}\langle h, v_{(1)}\rangle$, where $\delta(v) = v_{(0)} \otimes v_{(1)}$. Thus $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$ and the corresponding category of $\mathcal{O}_q(SL_2)$-comodules are two descriptions of essentially the same finite-dimensional representation theory.

Quantum Peter–Weyl

Classically, matrix coefficients of irreducible representations span the coordinate functions on a reductive group.

The quantum analogue remains true.

For generic q, let $V_n = V(n\omega)$, $n = 0, 1, 2, \dots$, denote the irreducible finite-dimensional $U_q(\mathfrak{sl}_2)$-modules.

Then, as a left-right $U_q(\mathfrak{sl}_2)$-module, or equivalently as an $\mathcal{O}_q(SL_2)$-bicomodule, one has the quantum Peter–Weyl decomposition

$$\mathcal{O}_q(SL_2) \cong \bigoplus_{n \geq 0} V_n^* \otimes V_n.$$

The meaning is exactly the classical one: each irreducible representation contributes its full matrix coefficient space.

For SL_2, all V_n occur in tensor powers of the fundamental module V_1. Hence the four coefficients a, b, c, d are enough to generate the coordinate algebra.

See Klimyk–Schmüdgen [Klimyk–Schmüdgen 1997, §4.2.5]. For Lusztig's much more refined version, involving the canonical basis, see [Lusztig 2010, Ch. 29, especially §§29.3–29.5].

This is worth mentioning even without proof because it describes the size of the object reconstructed by Tannaka theory.

Classical limit on the coordinate side

At $q = 1$, the quadratic relations become commutative and the quantum determinant becomes $ad - bc = 1$; hence $\mathcal{O}_q(SL_2)$ specializes, using the Laurent-polynomial integral-form procedure defined in Lecture I, to $\mathcal{O}(SL_2)$. This is the coordinate-side counterpart of the classical limit discussed there. The horizontal relation between the enveloping and coordinate sides remains Hopf-theoretic duality, not ordinary vector-space duality.

Reading guide

For this lecture the most efficient reading is:

(1) Kassel: Ch. VII, §4 for the duality between $U_q(\mathfrak{sl}_2)$ and $SL_q(2)$; §5 for modules and comodules; Ch. VIII, especially §§1, 5–7, for the material which will begin Lecture III.
(2) Klimyk–Schmüdgen: §4.1 for $\mathcal{O}(SL_q(2))$; §4.2 for its corepresentation theory and Peter–Weyl decomposition; §4.4 for the dual pairing with $U_q(\mathfrak{sl}_2)$. For the general coordinate-algebra construction see Ch. 9, pp. 303–330.
(3) Lusztig: Ch. 27 for tensor products of based modules and Ch. 29 for the refined Peter–Weyl theorem, especially §29.5 for the quantum coordinate algebra.
(4) Majid: Ch. 9, especially §9.4, for categorical Tannaka–Krein reconstruction.
(5) Deligne–Milne: Theorem 2.11 for the classical neutral Tannakian reconstruction theorem.

References

[Deligne–Milne 1982] P. Deligne and J. S. Milne, "Tannakian Categories," in *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer, Berlin, 1982, pp. 101–228. See especially Theorem 2.11.

[Etingof–Gelaki–Nikshych–Ostrik 2015] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, Mathematical Surveys and Monographs 205, American Mathematical Society, Providence, RI, 2015. See especially Ch. 5.

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[Kassel 1995] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics 155, Springer-Verlag, New York, 1995. For this lecture see especially Ch. VII, §4, pp. 150–154, and §5, pp. 154–155. For the R-matrix and FRT continuation see Ch. VIII.

[Klimyk–Schmüdgen 1997] A. Klimyk and K. Schmüdgen, *Quantum Groups and Their Representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin–Heidelberg, 1997. For this lecture see §4.1, pp. 97–104, §4.2, pp. 104–113, and §4.4, beginning at p. 119. For the general coordinate-algebra viewpoint see Ch. 9, pp. 303–330.

[Lusztig 2010] G. Lusztig, *Introduction to Quantum Groups*, Modern Birkhäuser Classics, Birkhäuser, Boston, 2010 (reprint of the 1993 edition, Progress in Mathematics 110). See especially Ch. 27 and Ch. 29, in particular §29.5.

[Majid 1995] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995. See Ch. 9, especially §9.4, for Tannaka–Krein reconstruction.

[Sweedler 1969] M. E. Sweedler, *Hopf Algebras*, W. A. Benjamin, New York, 1969.

[Waterhouse 1979] W. C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics 66, Springer-Verlag, New York, 1979. See especially §§1.1–1.5 for the equivalence between affine group schemes and commutative Hopf algebras.

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[Woronowicz 1988] S. L. Woronowicz, "Tannaka–Krein Duality for Compact Matrix Pseudogroups. Twisted $SU(N)$ Groups," *Inventiones Mathematicae* **93** (1988), 35–76.